Oxford Cambridge and RSA

Monday 26 June 2017 – Afternoon

A2 GCE MATHEMATICS

4726/01 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726/01
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found inside the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. If additional space is required, you should use the lined page(s) at the end of the Printed Answer Book. The question number(s) must be clearly shown.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **20** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER/INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Answer **all** the questions.

1 Find the x-coordinate of the stationary point on the curve $y = 3\cosh x - 2\sinh x$, giving your answer exactly in logarithmic form. **[3]**

2 By first completing the square in the denominator, find the exact value of

$$\int_3^4 \frac{1}{x^2 - 6x + 10}\,dx.$$ **[4]**

3 It is given that $y = \tan^{-1}\left(\frac{x}{x+1}\right)$.

(i) Show that, when $x = 0$, $\frac{d^2y}{dx^2} = -2$. **[4]**

(ii) Find the Maclaurin's series for $\tan^{-1}\left(\frac{x}{x+1}\right)$ up to and including the term in x^2. **[2]**

4 The equation of a curve is $y = \frac{x-1}{x^2 - x - 2}$.

(i) Write down the equations of the asymptotes. **[3]**

(ii) State the coordinates of the point where the curve cuts one of its asymptotes. **[1]**

(iii) Show that there are no turning points on the curve. **[4]**

(iv) Sketch the curve. **[2]**

5 The polar equations of two curves are $r = \sin\theta$ and $r = \cos 2\theta$.

(i) On the same diagram sketch the parts of both curves that are in the positive quadrant. **[4]**

The curves meet at the origin and also at a point A.

(ii) Determine the polar coordinates of A. **[2]**

(iii) Find the exact area between the two curves in the positive quadrant. **[8]**

6 **(i)** Using the definition of $\sinh x$ in terms of e^x and e^{-x}, show that

$$\sinh 3x = 4\sinh^3 x + 3\sinh x.$$ **[3]**

(ii) Use the substitution $w = \sinh x$ to find the real root of the equation

$$4w^3 + 3w - 3 = 0.$$

Give your answer in the form $a\ln\left(b+\sqrt{c}\right)$ where a, b and c are real numbers to be determined. **[4]**

7 It is given that $I_n = \int_0^1 x^n\sqrt{1-x}\,dx$ for $n \geqslant 0$.

(i) Show that $I_n = \frac{2n}{2n+3}I_{n-1}$. **[5]**

(ii) Deduce that $I_n < I_{n-1}$. **[2]**

(iii) Show that $I_4 = \frac{256}{3465}$. **[3]**

8 It is required to solve the equation $f(x) = \ln(4x-1) - x = 0$.

(i) Show that the equation has two roots, α and β, such that $0.5 < \alpha < 1$ and $1 < \beta < 2$. **[1]**

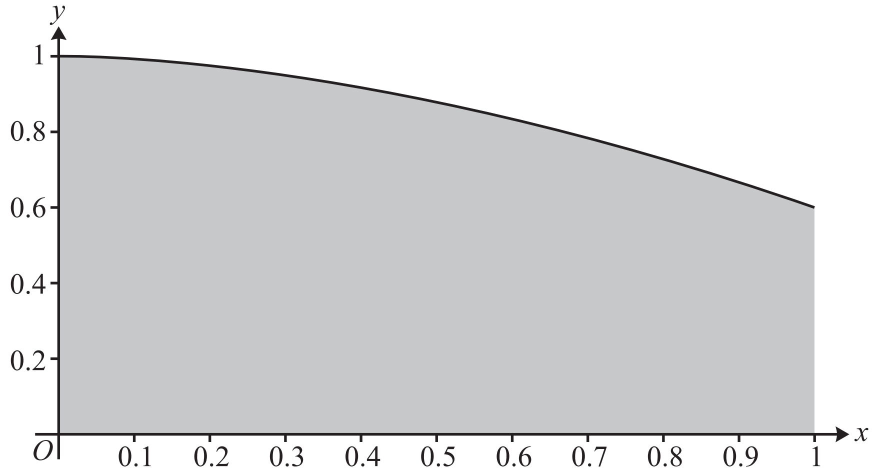
(ii) Use the iterative formula $x_{r+1} = \ln(4x_r - 1)$ with $x_0 = 1.8$ to find x_1, x_2 and x_3, correct to 5 decimal places. Write down the value of β to as many decimal places as these values justify . **[3]**

(iii) Derive the iterative formula $x_{r+1} = \frac{\left(e^{x_r}+1\right)}{4}$ and use it to find α correct to 4 decimal places. **[4]**

(iv) Show that the iterative formula in part **(ii)** will not find the value of α. Determine whether the iterative formula in part **(iii)** will find the value of β. **[3]**

Question 9 begins on page 4.

9 The diagram shows the curve $y = e^{-x^2}$ for $0 \leqslant x \leqslant 1$. The region between the curve and the x-axis for $0 < x < 1$ is shaded.



(i) By considering n rectangles of equal width, show that an upper bound, U, for the area of the shaded region is $U = \frac{1}{n}\sum_{r=0}^{n-1} e^{-\left(\frac{r}{n}\right)^2}$. **[3]**

(ii) By considering another set of n rectangles of equal width, find a similar expression for a lower bound, L, for the area of the shaded region. **[1]**

(iii) Determine the least value of n such that $U - L < 10^{-4}$. **[3]**

END OF QUESTION PAPER

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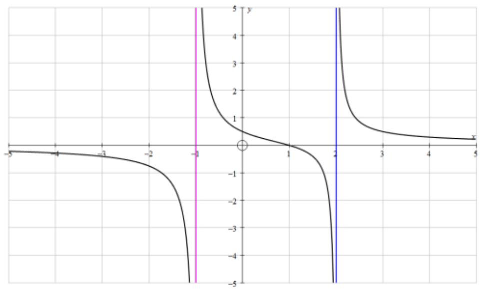
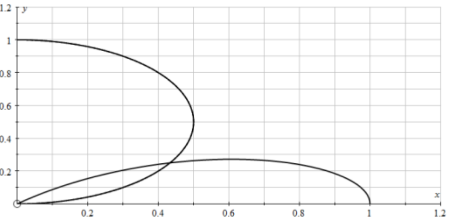
Question			Answer	Marks	Guidance	
1			$y = 3\cosh x - 2\sinh x \Rightarrow \frac{dy}{dx} = 3\sinh x - 2\cosh x$	M1	Correct diffn and **soi** set = 0	Alternatively: $y = \frac{1}{2}\left(e^x + 5e^{-x}\right)$
			$= 0 \Rightarrow \tanh x = \frac{2}{3}$	A1		$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(e^x - 5e^{-x}\right) = 0$ when $e^x = 5e^{-x}$
			$\Rightarrow x = \frac{1}{2}\ln\left(\frac{1+\frac{2}{3}}{1-\frac{2}{3}}\right) = \frac{1}{2}\ln 5$	A1	$= \ln\sqrt{5}$ www	$\Rightarrow e^{2x} = 5 \Rightarrow x = \frac{1}{2}\ln 5$ If y is wrong then M0
				[3]		

Question			Answer	Marks	Guidance	
2			$I = \int_3^4 \frac{1}{x^2 - 6x + 10}\,dx = \int_3^4 \frac{1}{(x-3)^2 + 1}\,dx$	M1 A1	Complete the square. Denominator	Sight of $(x \pm 3)^2 + 1$
			From formula book: $I = \left[\tan^{-1}(x-3)\right]_3^4 = \tan^{-1}1 \pm \tan^{-1}0$	M1	Use standard form and apply limits	Accept $\tan^{-1}(x \pm 3)$
			$= \frac{\pi}{4}$	A1	**www**	SC correct answer only B1
				[4]		

Question			Answer	Marks	Guidance	
3	(i)		$y=\tan^{-1}\left(\frac{x}{x+1}\right)\Rightarrow\frac{dy}{dx}=\frac{1}{1+\left(\frac{x}{x+1}\right)^2}\cdot\frac{(x+1).1-x.1}{(x+1)^2}$	M1	Attempt to apply chain rule and deal with quotient	$\tan y=\frac{x}{x+1}\Rightarrow\sec^2 y\frac{dy}{dx}=\frac{1}{(x+1)^2}$ M1 for implicit diffn A1 for expression for y'
			$=\frac{1}{(x+1)^2+x^2}=\frac{1}{2x^2+2x+1}$	A1	Soi	$\Rightarrow\sec^2 y\frac{d^2y}{dx^2}+2\sec^2 y\tan y\left(\frac{dy}{dx}\right)^2=-\frac{2}{(x+1)^3}$ A1
			$\Rightarrow\frac{d^2y}{dx^2}=-\frac{4x+2}{(2x^2+2x+1)^2}$	A1		Substitute $x=0,\ \tan y=0,\sec y=1$
			$(\text{When } x=0,)\ \frac{d^2y}{dx^2}=\left(-\frac{2}{(1)^2}\right)=-2$	A1	www AG	$\Rightarrow\frac{d^2y}{dx^2}=-2$ A1
				[4]		
	(ii)		$y_0=0,\ y_0'=1$ www in part (i) soi $\Rightarrow(y=)\,x-x^2$	DB1 B1ft	Use values from (i)	Dependent on correct working in (i) i.e. $y = 0 + kx - x^2$ where k is *their* y'
				[2]		

Question			Answer	Marks	Guidance	
4	(i)		$x = 2,\ x = -1,\ y = 0$	B3	B1 for each	
				[3]		
	(ii)		(1, 0)	B1	Allow $x = 1$, $y = 0$ but not just $x = 1$	No extras
				[1]		
	(iii)		$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(x^2-x-2\right)1-\left(x-1\right)\left(2x-1\right)}{\left(x^2-x-2\right)^2}$ $= 0$ when $x^2 - x - 2 - 2x^2 + 3x - 1 = 0$ $\Rightarrow x^2 - 2x + 3 = 0$ $"b^2 - 4ac" = 4 - 12 = -8 < 0$ so no (real) roots (so gradient fn $\neq 0$) Alternative $y = \frac{1}{3(x-2)} + \frac{2}{3(x+1)}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3(x-2)^2} - \frac{2}{3(x+1)^2} < 0$ for all values of x	M1 A1 A1 A1	Diffn dealing with quotient and set = 0 Or $(x-1)^2 = -2$ M1 A1 A1 A1www	As a quadratic in x: $x^2y - x(y+1) + (1-2y) = 0$ Coincident roots if $(y+1)^2 = 4y(1-2y)$ $\Rightarrow 9y^2 - 2y + 1 = 0$ This has no real roots So there is no value of y which gives coincident values of x. so no turning points. M1 A1 for writing as quadratic in x and setting up coincident roots $"b^2 - 4ac\left(=0\right)"$ A1 quadratic equation in y A1 no roots
				[4]		
	(iv)			B1 DB1	General shape, in three parts, gradient always negative Dep on 1st B Asymptotes shown and intercepts evident	If rotated then ok providing axes are labelled. i.e. x axis, halfway between 0 and 2 and y axis either a clear scale or point marked
				[2]		

Question			Answer	Marks	Guidance	
5	(i)			B1 B1 B1 DB1	first curve approximately correct second curve approx correct intersections on "axes" curves intersect approximately in the right place, dep on 1st two B marks	Ignore any extra parts of curves
				[4]		
	(ii)		$r = 0.5$ $\theta = \frac{\pi}{6}$	B1 B1	Accept $\left(0.5, \frac{\pi}{6}\right)$ or $\left(\frac{\pi}{6}, 0.5\right)$	
				[2]		

Question			Answer	Marks	Guidance	
	(iii)		Limit of $\frac{\pi}{6}$	B1	Seen anywhere in either integral	$\frac{1}{2}\int\left(r_1^2 - r_2^2\right)\mathrm{d}\theta$ could earn the first 6 marks
			For first: $A = \frac{1}{2}\int_0^{\pi/6} r^2\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/6}\sin^2\theta\,\mathrm{d}\theta = \frac{1}{4}\int_0^{\pi/6}(1-\cos 2\theta)\,\mathrm{d}\theta$	M1	Formula for area applied to at least one curve ignoring limits	
				DM1	Deal with $\sin^2\theta$ correctly	
			$=\frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta\right)$ $\left(=\frac{1}{4}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)\right)$	A1	answer before limits	
			For second: $A = \frac{1}{2}\int_{\pi/6}^{\pi/4} r^2\,\mathrm{d}\theta = \frac{1}{2}\int_{\pi/6}^{\pi/4}\cos^2 2\theta\,\mathrm{d}\theta = \frac{1}{4}\int_{\pi/6}^{\pi/4}(1+\cos 4\theta)\,\mathrm{d}\theta$	DM1	deal with $\cos^2 2\theta$ correctly	
			$=\frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)$ $\left(=\frac{\pi}{48}-\frac{\sqrt{3}}{32}\right)$	A1	answer before limits	
			Total area = $=\frac{\pi}{16}-\frac{3\sqrt{3}}{32}$	M1	Add *their* numerical areas from different limits (or subtract appropriately)	
				A1	cao	
				[8]		

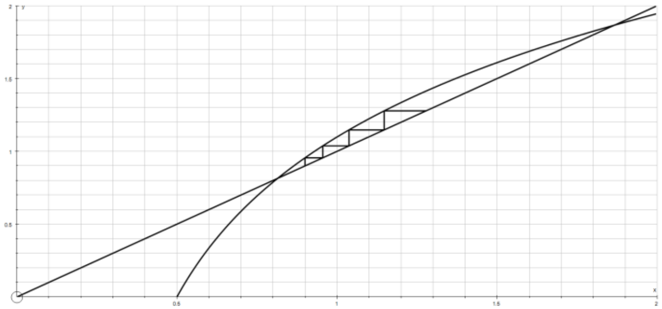
Question			Answer	Marks	Guidance	
6	(i)		$\sinh x = \frac{e^x - e^{-x}}{2}$, $\sinh 3x = \frac{e^{3x} - e^{-3x}}{2}$ $\sinh^3 x = \left(\frac{e^x - e^{-x}}{2}\right)^3 = \frac{1}{8}\left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right)$ $\Rightarrow 4\sinh^3 x = \left(\frac{e^{3x} - e^{-3x}}{2}\right) - 3\left(\frac{e^x - e^{-x}}{2}\right)$ $= \sinh 3x - 3\sinh x$ $\Rightarrow \sinh 3x = 4\sinh^3 x + 3\sinh x$	M1 A1 A1	Using exponentials for sinh*x* and expanding cubic Correct cubic **oe** Must include exponential form of sinh3*x*	
				[3]		
	(ii)		$4w^3 + 3w - 3 = 0$ $\Rightarrow 4\sinh^3 x + 3\sinh x - 3 = 0$ $\Rightarrow \sinh 3x = 3$ $\Rightarrow 3x = \ln\left(3 \pm \sqrt{10}\right)$ $\Rightarrow x = \frac{1}{3}\ln\left(3 + \sqrt{10}\right)$ $\Rightarrow \left(w = \sinh\left(\frac{1}{3}\ln\left(3 + \sqrt{10}\right)\right)\right)$	M1 M1 A1 A1	Make substitution Use result of (i) Must have plus sign. Ignore subsequent working	
				[4]		

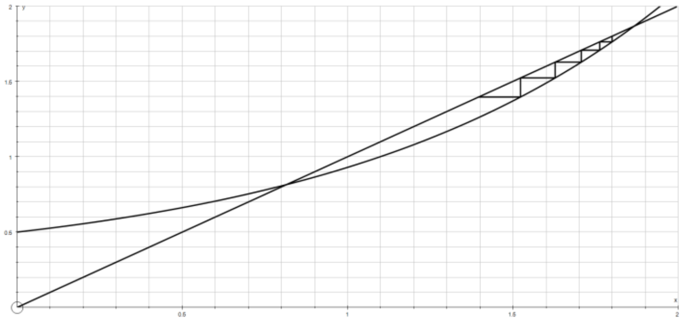
Question			Answer	Marks	Guidance	
7	(i)		$u = x^n \Rightarrow du = nx^{n-1}dx$			
			$dv = \sqrt{1-x}\,dx \Rightarrow v = -\frac{2}{3}(1-x)^{3/2}$	M1	Integrate by parts	
			$\Rightarrow I_n = \left[-x^n \frac{2}{3}(1-x)^{3/2}\right]_0^1 + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)^{3/2}\,dx$	A1 A1	1st term must be seen and later made = 0 Second term	
			$\Rightarrow I_n = 0 + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\,dx$ $= \frac{2}{3}n(I_{n-1} - I_n)$	M1	Dealing with $(1-x)^{3/2} = (1-x)\sqrt{1-x}$ And converting to I_n and I_{n-1}	
			$\Rightarrow \left(1 + \frac{2}{3}n\right)I_n = \frac{2n}{3}I_{n-1} \Rightarrow I_n = \frac{2n}{2n+3}I_{n-1}$	A1	AG	
				[5]		
	(ii)		$\frac{2n}{2n+3} < 1 \Rightarrow I_n < I_{n-1}$	M1 A1		
				[2]		
	(iii)		$I_4 = \frac{8}{11}I_3 = \frac{8}{11}.\frac{6}{9}I_2 = \frac{8}{11}.\frac{6}{9}.\frac{4}{7}I_1 = \frac{8}{11}.\frac{6}{9}.\frac{4}{7}.\frac{2}{5}I_0$	M1	For using reduction formula	$I_0 = \frac{2}{3},\ I_1 = \frac{4}{15},\ I_2 = \frac{16}{105},\ I_3 = \frac{32}{315}$ $\Rightarrow I_4 = \frac{256}{3465}$
			$I_0 = \int_0^1 \sqrt{1-x}\,dx = \left[-\frac{2}{3}(1-x)^{3/2}\right]_0^1 = \frac{2}{3}$	B1	For I_0	
			$\Rightarrow I_4 = \frac{8}{11}.\frac{6}{9}.\frac{4}{7}.\frac{2}{5}.\frac{2}{3} = \frac{256}{3465}$	A1	AG	
				[3]		

Question			Answer	Marks	Guidance	
8	(i)		f(0.5) = –0.5 f(1) = 0.099 f(2) = –0.054 So sign changes gives roots in the required ranges. **oe**	**B1**	Must contain three decimal values and conclusion	e.g. "crosses axis"
				[1]		
	(ii)		$x_1 = 1.82455$ $x_2 = 1.84026$ $x_3 = 1.85019$ $\beta = 1.9$ is all that can be justified	**B1** **B1** **B1**	x_1 x_2 and x_3 Subtract 1 mark for not 5 dp β	
				[3]		
	(iii)		$x = \ln(4x-1) \Rightarrow 4x-1 = \mathrm{e}^x$ $\Rightarrow 4x = \mathrm{e}^x + 1 \Rightarrow x = \dfrac{\mathrm{e}^x+1}{4}$ giving $x_{r+1} = \dfrac{\mathrm{e}^{x_r}+1}{4}$ 0.81 0.811977 0.811977 0.813089 0.813089 0.813716 0.813716 0.814069 $\Rightarrow \alpha = 0.8145$	**M1** **A1** **M1** **A1**	Clear attempt to rearrange formula Must include the "*r*"s AG Starting with any value (which must be seen) in range [0.5,1] Mark final answer	
				[4]		

Question		Answer	Marks	Guidance	
	(iv)	For first iterative formula: $g(x) = \ln(4x-1)$ $\Rightarrow g'(x) = \dfrac{4}{4x-1}$ $\lvert g'(\alpha)\rvert > 1$ (=1.77) so will not converge to α For 2nd iterative formula: $g(x) = \dfrac{e^x+1}{4}$ $\Rightarrow g'(x) = \dfrac{1}{4}e^x$ $\lvert g'(1.9)\rvert > 1$ (=1.67) so it won't converge to β	**M1** **A1** **A1**	For finding g '(x) in either case. Or $\lvert g'(x)\rvert < 1$ when $x > 1.25$ so no Accept value in range [0.8,0.9] Or $\lvert g'(x)\rvert < 1$ when $x < \ln 4 \approx 1.4$ so no Accept value in range [1.8,2]	SC Demonstrating divergence numerically or graphically for either using an initial value near the root on either side. M1 1st demonstration plus conclusion (at least 3 numeric iterations or 2 steps)A1 2nd demonstration plus conclusion (at least 3 numeric iterations or 2 steps) A1
			[3]		

Graphs





Question			Answer	Marks	Guidance	
9	(i)			M1	Evidence of using correct rectangles (may be by diagram) with heights the left hand side with a clear indication that they are all above the curve	SC: if n = 10 is used then M0 but B1 may be earned
			1st rectangle has area $\frac{1}{n}e^{0}$ 2[nd] rectangle has area $\frac{1}{n}e^{-\left(\frac{1}{n}\right)^2}$	B1	Correct width and height of at least one rectangle - do not accept $\frac{1}{n}e^{-\left(\frac{0}{n}\right)^2}$	
			Last rectangle has area $\frac{1}{n}e^{-\left(\frac{n-1}{n}\right)^2}$ Giving $U=\frac{1}{n}\sum_{r=0}^{n-1}e^{-\left(\frac{r}{n}\right)^2}$	A1	Considering last rectangle and sum giving complete solution www AG	
				[3]		
	(ii)		$L=\frac{1}{n}\sum_{r=1}^{n}e^{-\left(\frac{r}{n}\right)^2}$	B1		
				[1]		
	(iii)		$U-L=\frac{1}{n}\left(\sum_{r=0}^{n-1}e^{-\left(\frac{r}{n}\right)^2}-\sum_{r=1}^{n}e^{-\left(\frac{r}{n}\right)^2}\right)$ $=\frac{1}{n}\left(e^{-\left(\frac{0}{n}\right)^2}-e^{-\left(\frac{n}{n}\right)^2}\right)=\frac{1}{n}\left(1-e^{-1}\right)$	M1	Dealing with cancelling of terms	
			$U-L<10^{-4}$ $\Rightarrow\frac{1}{n}\left(1-e^{-1}\right)<10^{-4}$	A1	Correct inequality soi	
			$\Rightarrow n>\left(1-e^{-1}\right)\times10^{4}\Rightarrow n=6322$	A1		
				[3]		